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Decomposition of k^{th} Order Slant Toeplitz Operators

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Abstract

In this paper, we establish the block matrix decomposition of k^{th} order slant Toeplitz operators. We also establish some relations between the compressions of k^{th} order slant Toeplitz and k^{th} order slant Hankel operators on H^2. In the last section, we introduce the notion of k^{th} order slant Toeplitz graphs.

Keywords: generalized slant Toeplitz operator; generalized slant Hankel operator; block matrix decomposition; k^{th} order slant Toeplitz graphs.

1 Introduction

In this paper, we will denote by $\mathbb{T}$ the unit circle, $L^2(\mathbb{T})$ the Hilbert space of square-integrable functions on the unit circle $\mathbb{T}$, $L^\infty(\mathbb{T})$ the space of all essentially measurable functions on $\mathbb{T}$, $H^2(\mathbb{T})$ the subspace of $L^2(\mathbb{T})$ of analytic functions on $\mathbb{T}$ and $H^\infty(\mathbb{T})$ the space of all the functions that are analytic and bounded on $\mathbb{T}$.

Otto Toeplitz introduced and studied the Toeplitz operators in the year 1911 [13]. The relationships between the matrices and symbols of Laurent and Toeplitz matrices were established by Brown and Halmos [4]. Many properties of Toeplitz operators have been further established. The study of slant Toeplitz operators is due to the efforts of Ho in 1996 [9]. Some basic properties such as norm, spectrum, compactness, eigenvalues and eigenvectors, etc., of this type of operators were also extensively studied [9].

The notion of slant Hankel operator was introduced by Arora et al. in 2006 [2]. Several basic properties such as norm, compactness along with various spectral properties were also discussed [1]. The characterization of a k^{th} order slant Hankel operator and its several algebraic properties were analysed by Arora and Bhola [3]. Further, the relations between Hankel and Toeplitz operators via block matric decomposition of multiplication operators were established by Chu [5] to study the compactness of their product.

In this paper, we try to extend the idea of this block matrix decomposition on generalized slant Toeplitz operators on L^2 to establish various relations between the compressions of generalized slant Hankel and generalized slant Toeplitz operators on H^2.

2 k^{th} Order Slant Toeplitz Operators

Let $\mathcal{B} = \{z^i \mid i \in \mathbb{Z}\}$ be the standard basis for $L^2(\mathbb{T})$. If $\varphi \in L^\infty(\mathbb{T})$ is a function, then $\varphi(z)$ is of the form given by

$$\varphi(z) = \sum_{i=-\infty}^{\infty} a_i z^i,$$

where $a_i = \langle \varphi, z^i \rangle$ is the i^{th} Fourier coefficient of φ.

Definition 2.1. *The Toeplitz operator $T_\varphi : H^2 \longrightarrow H^2$ is defined by $T_\varphi(f) = P(\varphi f)$, for all $f \in H^2$.*

Definition 2.2 (k^{th} order slant Toeplitz operator)**.** *For any integer $k \geq 2$, the k^{th} order slant Toeplitz operator U_φ^k on $L^2(\mathbb{T})$ is the operator given by,*

$$U_\varphi^k(z^l) = \sum_{i=-\infty}^{\infty} a_{ki-l} z^i.$$

With respect to the standard basis $\mathcal{B}$, the matrix representation of U_φ^k is the given by,

$$[U_\varphi^k]_{\mathcal{B}} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & a_0 & a_{-1} & a_{-2} & \cdots & a_{-k} & \cdots \\ \cdots & a_k & a_{k-1} & a_{k-2} & \cdots & a_0 & \cdots \\ \cdots & a_{2k} & a_{2k-1} & a_{2k-2} & \cdots & a_k & \cdots \\ \cdots & a_{3k} & a_{3k-1} & a_{3k-2} & \cdots & a_{2k} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Definition 2.3. *Now, the compression V_φ^k of U_φ^k to the space $H^2(\mathbb{T})$ is defined by,*

$$V_\varphi^k = PU_\varphi^k\big|_{H^2},$$

where P is the orthogonal projection of L^2 onto H^2.

Then, the matrix of V_φ^k is given by,

$$[V_\varphi^k] = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdots & a_{-k} & \cdots \\ a_k & a_{k-1} & a_{k-2} & \cdots & a_0 & \cdots \\ a_{2k} & a_{2k-1} & a_{2k-2} & \cdots & a_k & \cdots \\ a_{3k} & a_{3k-1} & a_{3k-2} & \cdots & a_{2k} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Definition 2.4. *Let W_k be the operator on $L^2(\mathbb{T})$ defined by,*

$$W_k(z^i) = \begin{cases} z^{\frac{i}{k}} & \text{if } k \mid i, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.5. *The adjoint of W_k is given by,*

$$W_k^* z^n = z^{nk}, \text{ for all } n \in \mathbb{Z}.$$

For all nonnegative integers n and for $r = 1, 2, 3, \ldots, k-1$, we have

$$PW_k z^{kn} = Pz^n = z^n = W_k z^{kn} = W_k P z^{kn},$$

and

$$PW_k z^{kn+r} = P0 = 0 = W_k z^{kn+r} = W_k P z^{kn+r}.$$

Also,

$$PW_k^* z^n = Pz^{kn} = z^{kn} = W_k^* z^n = W_k^* P z^n.$$

It is obvious that $U_\varphi^k = W_k M_\varphi$, where M_φ is the multiplication operator on L^2 induced by φ. Now,

$$V_\varphi^k = PU_\varphi^k\big|_{H^2} = PW_k M_\varphi\big|_{H^2} = W_k P M_\varphi\big|_{H^2} = W_k T_\varphi,$$

where $T_\varphi = PM_\varphi\big|_{H^2}$ is a Toeplitz operator on $H^2(\mathbb{T})$.

Definition 2.6. *The Hankel operator $H_\varphi : H^2 \longrightarrow H^2$ is defined by,*

$$H_\varphi(f) = PJ(\varphi f), \text{ for all } f \in H^2,$$

where J denotes the flip operator on L^2 given by $J(f(z)) = f(\bar{z})$ for all $f \in L^2$.

Definition 2.7 (k^{th} order slant Hankel operator)**.** *For any integer $k \geq 2$, the k^{th} order slant Hankel operator D_φ^k on $L^2(\mathbb{T})$ is the operator given by,*

$$D_\varphi^k(z^l) = \sum_{i=-\infty}^{\infty} a_{-ki-l} z^i.$$

With respect to the standard basis $\mathcal{B}$, the matrix representation of the k^{th} order slant Hankel operator D_φ^k is given by,

$$\left[D_\varphi^k\right]_{\mathcal{B}} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & a_0 & a_{-1} & a_{-2} & \cdots & a_{-k} & \cdots \\ \cdots & a_{-k} & a_{-k-1} & a_{-k-2} & \cdots & a_{-2k} & \cdots \\ \cdots & a_{-2k} & a_{-2k-1} & a_{-2k-2} & \cdots & a_{-3k} & \cdots \\ \cdots & a_{-3k} & a_{-3k-1} & a_{-3k-2} & \cdots & a_{-4k} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Definition 2.8. *Now, the compression E_φ^k of D_φ^k to the space $H^2(\mathbb{T})$ is given by,*

$$E_\varphi^k = PD_\varphi^k\big|_{H^2},$$

where P is the orthogonal projection of L^2 onto H^2.

Then, the matrix of E_φ^k is given by,

$$\left[E_\varphi^k\right] = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdots & a_{-k} & \cdots \\ a_{-k} & a_{-k-1} & a_{-k-2} & \cdots & a_{-2k} & \cdots \\ a_{-2k} & a_{-2k-1} & a_{-2k-2} & \cdots & a_{-3k} & \cdots \\ a_{-3k} & a_{-3k-1} & a_{-3k-2} & \cdots & a_{-4k} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Obviously, $D_\varphi^k = JW_kM_\varphi$.

Thus,

$$E_\varphi^k = PD_\varphi^k\big|_{H^2} = PJW_kM_\varphi\big|_{H^2} = W_kPJM_\varphi|_{H^2} = W_kH_\varphi,$$

where $H_\varphi = PJM_\varphi\big|_{H^2}$ is a Hankel operator on $H^2(\mathbb{T})$.

Lemma 2.1. *The adjoint of W_k is multiplicative on $L^2(\mathbb{T})$, i.e., for any $\varphi, \psi \in L^2(\mathbb{T})$, we have $W_k^*(\varphi\psi) = (W_k^*\varphi)(W_k^*\psi)$.*

Proof. It can be easily seen that,

$$W_k^*(\varphi\psi) = (\varphi\psi)(z^k) = \varphi(z^k)\cdot\psi(z^k) = (W_k^*\varphi)(W_k^*\psi).$$

□

Theorem 2.1. *If $\varphi, \psi \in L^2(\mathbb{T})$, then*

$$W_k(\varphi\psi) = (W_k\varphi)(W_k\psi) + \sum_{r=1}^{k-1} z(W_k\bar{z}^{\,r}\varphi)(W_k\bar{z}^{\,k-r}\psi).$$

Proof. Let P_r denote the projection from L^2 on the closed span of $\{z^{kn+r} : n \in \mathbb{Z} \text{ and } r = 0, 1, 2, \ldots, k-1\}$. Then, $\varphi \in L^2(\mathbb{T})$ can be written as

$$\varphi(z) = \varphi_0(z^k) + \varphi_1(z^{k+1}) + \varphi_2(z^{k+2}) + \varphi_3(z^{k+3}) + \cdots + \varphi_{k-1}(z^{k+k-1}),$$

where $\varphi_r(z^k) = P_r(\varphi(z))$. Now,

$$\varphi(z) = \varphi_0(z^k) + z\varphi_1(z^k) + z^2\varphi_2(z^k) + z^3\varphi_3(z^k) + \cdots + z^{k-1}\varphi_{k-1}(z^k) = \sum_{r=0}^{k-1} z^r\varphi_r(z^k).$$

Similarly, $\psi \in L^2(\mathbb{T})$ can also be written as,

$$\psi(z) = \psi_0(z^k) + z\psi_1(z^k) + z^2\psi_2(z^k) + z^3\psi_3(z^k) + \cdots + z^{k-1}\psi_{k-1}(z^k) = \sum_{s=0}^{k-1} z^s\psi_s(z^k).$$

Now,

$$\begin{aligned}
\varphi(z)\psi(z) &= \left(\sum_{r=0}^{k-1} z^r\varphi_r(z^k)\right)\left(\sum_{s=0}^{k-1} z^s\psi_s(z^k)\right)\\
&= \sum_{r=0}^{k-1}\sum_{s=0}^{k-1} z^{r+s}\varphi_r(z^k)\psi_s(z^k)\\
&= \varphi_0(z^k)\psi_0(z^k) + \sum_{r=1}^{k-1}\sum_{s=1}^{k-1} z^{r+s}\varphi_r(z^k)\psi_s(z^k).
\end{aligned}$$

It can be seen that,

$$\begin{aligned}
W_k\left(\varphi(z)\psi(z)\right) &= W_k\left(\varphi_0(z^k)\psi_0(z^k)\right) + W_k\left(\sum_{r=1}^{k-1}\sum_{s=1}^{k-1} z^{r+s}\varphi_r(z^k)\psi_s(z^k)\right)\\
&= W_k\left(\varphi_0(z^k)\psi_0(z^k)\right) + W_k\left(\sum_{r,s=1}^{k-1} z^{r+s}\varphi_r(z^k)\psi_s(z^k)\right)\\
&= W_k\left(\varphi_0(z^k)\psi_0(z^k)\right) + W_k\left(\sum_{r=1}^{k-1} z^k\varphi_r(z^k)\psi_{k-r}(z^k)\right)\\
&= W_k\left(W_k^*(\varphi_0(z)\psi_0(z))\right) + W_k\left(\sum_{r=1}^{k-1} W_k^*(z\varphi_r(z)\psi_{k-r}(z))\right)\\
&= \varphi_0(z)\psi_0(z) + \sum_{r=1}^{k-1} z\varphi_r(z)\psi_{k-r}(z)\\
&= \left(W_k\varphi(z)\right)\left(W_k\psi(z)\right) + z\sum_{r=1}^{k-1} W_k\left(\bar{z}\varphi(z)\right)W_k\left(\bar{z}^{(k-r)}\psi(z)\right).
\end{aligned}$$

Hence, we have

$$W_k(\varphi\psi) = \left(W_k\varphi(z)\right)\left(W_k\psi(z)\right) + z\sum_{r=1}^{k-1} W_k\left(\bar{z}\varphi(z)\right)W_k\left(\bar{z}^{(k-r)}\psi(z)\right).$$

This proves the theorem. $\square$

Corollary 2.1. *If* $\varphi(z) = \sum_{n=-\infty}^{\infty} z^{kn}$ *or* $\psi(z) = \sum_{m=-\infty}^{\infty} z^{km}$, *then* W_k *is multiplicative on* $L^2(\mathbb{T})$ *i.e.,* $W_k(\varphi\psi) = (W_k\varphi)(W_k\psi)$.

Proof. If $\varphi(z) = \sum_{n=-\infty}^{\infty} z^{kn}$, then $W_k(\bar{z}^r\varphi) = 0$ for $r = 1, 2, \ldots, k-1$. Thus, by Theorem 2.1, we get

$$W_k(\varphi\psi) = (W_k\varphi)(W_k\psi).$$

Similarly, the result holds if $\psi(z) = \sum_{m=-\infty}^{\infty} z^{km}$. □

Corollary 2.2. *If* $\varphi(z) = \sum_{n=-\infty}^{\infty} z^{kn}$ *or* $\psi(z) = \sum_{m=-\infty}^{\infty} z^{km}$, *then the product of two* k^{th} *order slant Toeplitz operators is a* k^{th} *order slant Toeplitz operator i.e.,* $U^k_{\varphi\psi} = U^k_\varphi U^k_\psi$.

Proof. Since $U^k_\varphi = W_k M_\varphi$, we have $U^k_{\varphi\psi} f = W_k(\varphi\psi)f$ for all $f \in L^2(\mathbb{T})$.

If $\varphi(z) = \sum_{n=-\infty}^{\infty} z^{kn}$ or $\psi(z) = \sum_{m=-\infty}^{\infty} z^{km}$, then $U^k_{\varphi\psi} = (W_k\varphi)(W_k\psi)$.

Thus, $U^k_{\varphi\psi} = U^k_\varphi U^k_\psi$.

This proves the corollary. □

3 Block Matrix Representation of U^k_φ

A decomposition of a Hilbert space H in the form $M \oplus M^\perp$ leads to a block matrix representation of operators on H [12]. Suppose A is an operator on H. If P is the projection of H onto M, A_1 is the restriction of PA to M, A_2 is the restriction of PA to $M^\perp$, A_3 is the restriction of $(I-P)A$ to M and A_4 is the restriction of $(I-P)A$ to $M^\perp$, then A can be represented by

$$A = \left(\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right).$$

Since U^k_φ is an operator on $L^2(\mathbb{T})$, it can be expressed as an operator matrix with respect to the decomposition $L^2 = H^2 \oplus (H^2)^\perp$ as follows,

$$U^k_\varphi = \begin{pmatrix} PU^k_\varphi\big|_{H^2} & PU^k_\varphi\big|_{(H^2)^\perp} \\ (I-P)U^k_\varphi\big|_{H^2} & (I-P)U^k_\varphi\big|_{(H^2)^\perp} \end{pmatrix}.$$

If $\tilde{f}(z) = f(\bar{z})$, then the following relations can be easily established;

1. $PU^k_\varphi\big|_{H^2} = V^k_\varphi$.

2. $PU^k_{\varphi}|_{(H^2)^{\perp}} = E^k_{\widetilde{\varphi}}J$.
3. $(I-P)U^k_{\varphi}|_{H^2} = JE^k_{\varphi}$.
4. $(I-P)U^k_{\varphi}|_{(H^2)^{\perp}} = JV^k_{\widetilde{\varphi}}J$.

Thus, the block matrix representation of the k^{th} order slant Toeplitz operator $U^k_{\varphi} : L^2 \to L^2$ with respect to the decomposition $L^2 = H^2 \oplus (H^2)^{\perp}$ is given below,

$$U^k_{\varphi} = \begin{pmatrix} V^k_{\varphi} & E^k_{\widetilde{\varphi}}J \\ JE^k_{\varphi} & JV^k_{\widetilde{\varphi}}J \end{pmatrix}.$$

Similarly, for $\psi \in L^{\infty}(\mathbb{T})$, we have

$$U^k_{\psi} = \begin{pmatrix} V^k_{\psi} & E^k_{\widetilde{\psi}}J \\ JE^k_{\psi} & JV^k_{\widetilde{\psi}}J \end{pmatrix}.$$

Now,

$$\begin{aligned} U^k_{\varphi}U^k_{\psi} &= \begin{pmatrix} V^k_{\varphi} & E^k_{\widetilde{\varphi}}J \\ JE^k_{\varphi} & JV^k_{\widetilde{\varphi}}J \end{pmatrix} \begin{pmatrix} V^k_{\psi} & E^k_{\widetilde{\psi}}J \\ JE^k_{\psi} & JV^k_{\widetilde{\psi}}J \end{pmatrix} \\ &= \begin{pmatrix} V^k_{\varphi}V^k_{\psi} + E^k_{\widetilde{\varphi}}E^k_{\psi} & V^k_{\varphi}E^k_{\widetilde{\psi}}J + E^k_{\widetilde{\varphi}}V^k_{\widetilde{\psi}}J \\ JE^k_{\varphi}V^k_{\psi} + JV^k_{\widetilde{\varphi}}E^k_{\psi} & JE^k_{\varphi}E^k_{\widetilde{\psi}}J + JV^k_{\widetilde{\varphi}}V^k_{\widetilde{\psi}}J \end{pmatrix}. \end{aligned}$$

Also,

$$U^k_{\varphi\psi} = \begin{pmatrix} V^k_{\varphi\psi} & E^k_{\widetilde{\varphi\psi}}J \\ JE^k_{\varphi\psi} & JV^k_{\widetilde{\varphi\psi}}J \end{pmatrix}.$$

If $\varphi(z) = \displaystyle\sum_{n=-\infty}^{\infty} z^{kn}$ or $\psi(z) = \displaystyle\sum_{m=-\infty}^{\infty} z^{km}$, then, by Corollary 2.2, $U^k_{\varphi\psi} = U^k_{\varphi}U^k_{\psi}$.

Comparing the upper and lower left corners of the matrices $U^k_{\varphi\psi}$ and $U^k_{\varphi}U^k_{\psi}$, we get the following important relations between the compressions of k^{th} order slant Toeplitz and k^{th} order slant Hankel operators.

Theorem 3.1. *Let $\varphi, \psi \in L^{\infty}(\mathbb{T})$. If $\varphi(z) = \displaystyle\sum_{n=-\infty}^{\infty} z^{kn}$ or $\psi(z) = \displaystyle\sum_{m=-\infty}^{\infty} z^{km}$, then;*

(1) $V^k_{\varphi\psi} = V^k_{\varphi}V^k_{\psi} + E^k_{\widetilde{\varphi}}E^k_{\psi}$, and

(2) $E^k_{\varphi\psi} = E^k_{\varphi}V^k_{\psi} + V^k_{\widetilde{\varphi}}E^k_{\psi}$.

Corollary 3.1. *If $\psi \in zH^\infty$, then;*

(1) *the product of two compressions of k^{th} order slant Toeplitz operators on H^2 is a compression of a k^{th} order slant Toeplitz operator on H^2 i.e., $V^k_\varphi V^k_\psi = V^k_{\varphi\psi}$, and*

(2) *the product of a compression of a k^{th} order slant Hankel operator and a compression of a k^{th} order slant Toeplitz operator on H^2 is a compression of a k^{th} order slant Hankel operator i.e., $V^k_\varphi E^k_\psi = E^k_{\varphi\psi}$.*

Proof. If $\psi \in zH^\infty$, then,

$$\psi(z) = z\sum_{k=0}^{\infty} a_k z^k = \sum_{k=0}^{\infty} a_k z^{k+1}.$$

Thus,

$$J\psi(z) = J\left(\sum_{k=0}^{\infty} a_k z^{k+1}\right) = a_0 z^{-1} + a_1 z^{-2} + a_2 z^{-3} + \cdots \in (H^2)^\perp.$$

This implies that $H_\psi = PJ\psi = 0$, and hence, $E^k_\psi = 0$. Thus, from the above Theorem 3.1, we can obtain the following relations.

$$V^k_{\varphi\psi} = V^k_\varphi V^k_\psi \text{ and } E^k_{\varphi\psi} = E^k_\varphi V^k_\psi.$$

This proves the corollary. □

Theorem 3.2. *If $\widetilde{\varphi} \in zH^\infty$, then $V^k_\varphi E^k_\psi = E^k_\psi V^k_{\widetilde{\varphi}}$.*

Proof. By Theorem 3.1, we have

$$E^k_{\varphi\psi} = E^k_\varphi V^k_\psi + V^k_{\widetilde{\varphi}} E^k_\psi.$$

So,

$$E^k_{\widetilde{\varphi}\psi} = E^k_{\widetilde{\varphi}} V^k_\psi + V^k_\varphi E^k_\psi.$$

Since $E^k_{\widetilde{\psi}} = 0$ for $\widetilde{\varphi} \in zH^\infty$, we have

$$E^k_{\widetilde{\varphi}\psi} = V^k_\varphi E^k_\psi. \tag{1}$$

Similarly, $E^k_{\psi\varphi} = E^k_\psi V^k_\varphi + V^k_{\widetilde{\psi}} E^k_\varphi$ and $E^k_{\psi\widetilde{\varphi}} = E^k_\psi V^k_{\widetilde{\varphi}} + V^k_{\widetilde{\psi}} E^k_{\widetilde{\varphi}}$.
Hence,

$$E^k_{\widetilde{\varphi}\psi} = E^k_\psi V^k_{\widetilde{\varphi}}. \tag{2}$$

Thus, from (1) and (2), we get $V^k_\varphi E^k_\psi = E^k_\psi V^k_{\widetilde{\varphi}}$. □

4 k^{th} Order Slant Toeplitz Graphs

In [14], the concept of Toeplitz graphs with symmetric Toeplitz adjacency matrices had been studied. The corresponding graphs thus obtained were undirected simple graphs. But, what will be the nature of the graphs if we take non-symmetric adjacency Toeplitz matrices? To solve the discrepancy, the concept of directed Toeplitz graphs has been introduced in [11], in which the graphs are directed graphs without loops with Toeplitz adjacency matrices. In this paper, we extend the idea of these directed graphs with loops to introduce the notion of k^{th} order slant Toeplitz graphs whose adjacency matrices are k^{th} order slant Toeplitz matrices.

Definition 4.1. *A directed k^{th} order slant Toeplitz graph U^k is defined as a digraph with a k^{th} order slant Toeplitz adjacency matrix.*

The main diagonal of a k^{th} order slant Toeplitz adjacency matrix of order $(n \times n)$, will be labelled 0 and it contains only zeros. The $n-1$ distinct diagonals above the main diagonal will be labelled $1, 2, \dots, n-1$, and those under the main diagonal will also be labelled $1, 2, \dots, n-1$. Let $a_1, a_2, \dots, a_s$ be the upper diagonals containing ones and $b_1, b_2, \dots, b_t$ be the lower diagonals containing ones, such that $0 < a_1 < a_2 < \dots < a_s < n$ and $0 < b_1 < b_2 < \dots < b_t < n$. Then the corresponding k^{th} order slant Toeplitz graph will be denoted by $U_n^k\langle a_1, a_2, \dots, a_s; b_1, b_2, \dots, b_t\rangle$. Hence the graph $U_n^k\langle a_1, a_2, \dots, a_s; b_1, b_2, \dots, b_t\rangle$ with vertex set $\{1, 2, \dots, n\}$ is a digraph whose adjacency matrix is a k^{th} order slant Toeplitz matrix in which the arc (i, j) occurs if and only if $j = k(i-1) + 1 + a_r$ or $i = \dfrac{j-1}{k} + 1 + b_l$ for some $1 \le r \le s$ and $1 \le l \le t$.

Example 4.1. *Let $\varphi(z) = z^{-5} + 4z^{-4} + 3z^2 + 5z^4$.*

Then, by Definition 2.3, the matrix representation of V_φ^2 with respect to the orthonormal basis $\mathcal{B}$ is given by,

$$[V_\varphi^2]_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & 0 & 4 & 1 & 0 & 0 & 0 & \dots \\ 3 & 0 & 0 & 0 & 0 & 4 & 1 & 0 & \dots \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 4 & \dots \\ 5 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & \dots \\ 0 & 0 & 5 & 0 & 0 & 0 & 3 & 0 & \dots \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The indicator binary matrix (which replaces each non-zero entry by 1) of the above matrix is given by,

$$[V_\varphi^2]_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & \dots \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The corresponding k^{th} order slant Toeplitz graph of the above matrix will be $U^2_\infty\langle 3,4;1,3\rangle$ as given in Figure 1.

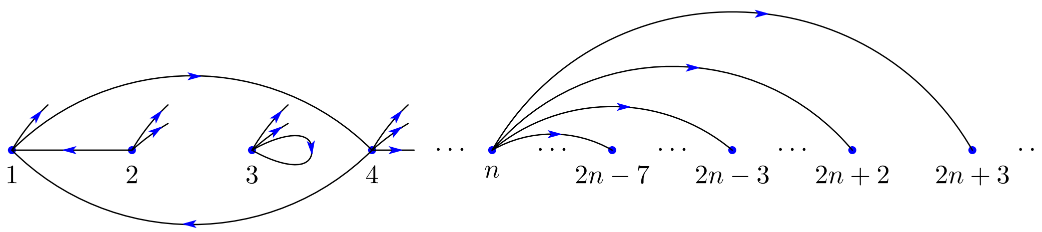

Figure 1: $U^2_\infty\langle 3,4;1,3\rangle$.

It may be noted that the out-degree of a vertex v is the number of arcs incident from v, and is denoted by $\mathrm{outdeg}(v)$ [15]. Similarly, the in-degree of a vertex v is the number of arcs incident to v, and is denoted by $\mathrm{indeg}(v)$. And if the digraph has loops, then each loop contributes 1 to both the out-degree and the in-degree of the corresponding vertex. Thus, for the given digraph $U^2_\infty\langle 3,4;1,3\rangle$ as shown in Figure 1, $\mathrm{outdeg}(1)=2$, $\mathrm{outdeg}(2)=3$, $\mathrm{outdeg}(3)=3$, $\mathrm{outdeg}(4)=4$, and $\mathrm{indeg}(1)=2$, $\mathrm{indeg}(2)=0$, $\mathrm{indeg}(3)=1$, $\mathrm{indeg}(4)=1$.

For an arbitrary vertex n, we can determine its out-degree by taking $i=n$ in the formulae $j=k(i-1)+1+a_r$ or $i=\dfrac{j-1}{k}+1+b_l$. In the above graph $U^2_\infty\langle 3,4;1,3\rangle$, we have $k=2$, $a_1=3$, $a_2=4$, $b_1=1$ and $b_2=3$ and thus we get the vertices incident from n given by $j=2n-7, 2n-3, 2n+2, 2n+3$. Clearly, for $n\geq 4$, the out-degree of the vertex n is 4.

Similarly, the vertices incident to n are given by $i=\dfrac{n-3}{2},\dfrac{n-2}{2},\dfrac{n+3}{2},\dfrac{n+7}{2}$. Thus, we can easily observe that for $n\geq 4$,

$$\mathrm{indeg}(n)=\begin{cases}1, & \text{if } n \text{ is even},\\ 3, & \text{if } n \text{ is odd}.\end{cases}$$

For the digraph $U^2_\infty\langle 3,4;1,3\rangle$, the out-degree and the in-degree sequences are $(2,3,3,4,4,4,\ldots)$ and $(0,1,1,\ldots,1,1,2,3,3,\ldots)$ respectively.

Example 4.2. *We can also find some digraphs which contain no loops. Consider a* 9×9 *upper triangular* k^{th} *order slant Toeplitz matrix with* $k=2$ *as given below:*

$$\begin{pmatrix}0&1&0&0&1&0&0&0&0\\0&0&0&1&0&0&1&0&0\\0&0&0&0&0&1&0&0&1\\0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0\end{pmatrix}.$$

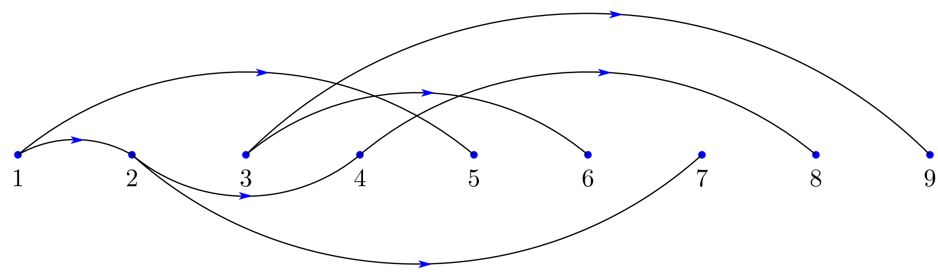

Figure 2: $U_9^2\langle 1, 5; 0\rangle$.

The corresponding k^{th} order slant Toeplitz graph of the above matrix will be $U_9^2\langle 1, 5; 0\rangle$ as given in Figure 2.

It can be clearly seen that $\text{outdeg}(1) = 2$, $\text{outdeg}(2) = 2$, $\text{outdeg}(3) = 2$, $\text{outdeg}(4) = 1$, $\text{outdeg}(5) = 0$, $\text{outdeg}(6) = 0$, $\text{outdeg}(7) = 0$, $\text{outdeg}(8) = 0$, $\text{outdeg}(9) = 0$, and $\text{indeg}(1) = 0$, $\text{indeg}(2) = 1$, $\text{indeg}(3) = 0$, $\text{indeg}(4) = 0$, $\text{indeg}(5) = 1$, $\text{indeg}(6) = 1$, $\text{indeg}(7) = 1$, $\text{indeg}(8) = 1$, $\text{indeg}(9) = 1$. Thus, for the digraph $U_9^2\langle 1, 5; 0\rangle$, the out-degree and in-degree sequences are $(0, 0, 0, 0, 0, 1, 2, 2, 2)$ and $(0, 0, 0, 1, 1, 1, 1, 1, 1)$ respectively.

Example 4.3. *By using the same function as in Example 4.1 and taking $k = 3$, the indicator binary matrix of V_φ^3 with respect to the orthonormal basis $\mathcal{B}$ is given by,*

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The corresponding k^{th} order slant Toeplitz graph of the above matrix will be $U_\infty^3\langle 3, 4; 1, 3\rangle$ as shown in Figure 3.

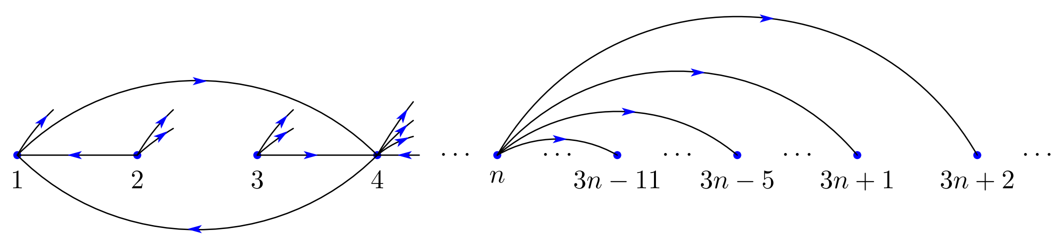

Figure 3: $U_\infty^3\langle 3, 4; 1, 3\rangle$.

It may be noted that $\text{outdeg}(1) = 2$, $\text{outdeg}(2) = 3$, $\text{outdeg}(3) = 3$, $\text{outdeg}(4) = 4$, and $\text{indeg}(1) = 2$, $\text{indeg}(2) = 0$, $\text{indeg}(3) = 0$, $\text{indeg}(4) = 3$. Now, considering $k = 3$, $a_1 = 3$, $a_2 = 4$, $b_1 = 1$ and $b_2 = 3$ and we get the vertices incident from an arbitrary vertex n given by $j = 3n - 11, 3n - 5, 3n + 1, 3n + 2$. Clearly, for $n \geq 4$, the out-degree of the vertex n is 4.

Similarly, the vertices incident to n are given by $i = \dfrac{n-1}{3}, \dfrac{n-2}{3}, \dfrac{n+5}{3}, \dfrac{n+11}{3}$. Thus, it can be seen that for $n \geq 4$,

$$\operatorname{indeg}(n) = \begin{cases} 0, & \text{if } n \equiv 0 \pmod 3, \\ 3, & \text{if } n \equiv 1 \pmod 3, \\ 1, & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Thus, the out-degree and the in-degree sequences for the digraph $U^3_\infty\langle 3,4;1,3\rangle$ are $(2,3,3,4,4,4,\dots)$ and $(0,\dots,0,1,1,\dots,1,1,2,3,3,\dots)$ respectively.

Example 4.4. *Taking $k = 4$ and using the function used in Example 4.1, the indicator binary matrix of V^4_φ with respect to the orthonormal basis $\mathcal{B}$ is given by,*

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The corresponding k^{th} order slant Toeplitz graph of the above matrix will be $U^4_\infty\langle 3,4;1,3\rangle$ as shown in Figure 4.

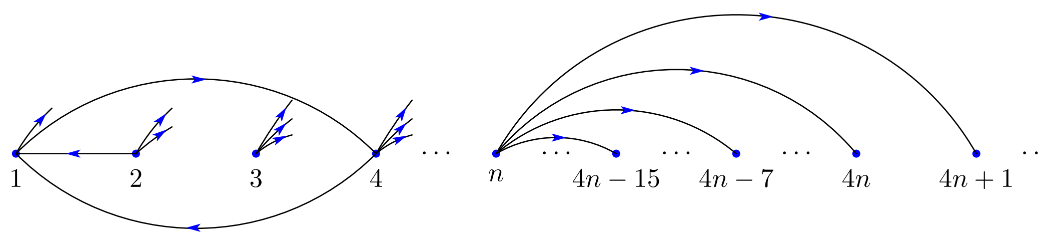

Figure 4: $U^4_\infty\langle 3,4;1,3\rangle$.

It may be noted that $\operatorname{outdeg}(1) = 2$, $\operatorname{outdeg}(2) = 3$, $\operatorname{outdeg}(3) = 3$, $\operatorname{outdeg}(4) = 4$, and $\operatorname{indeg}(1) = 2$, $\operatorname{indeg}(2) = 0$, $\operatorname{indeg}(3) = 0$, $\operatorname{indeg}(4) = 1$. Now, considering $k = 3$, $a_1 = 3$, $a_2 = 4$, $b_1 = 1$ and $b_2 = 3$ and we get the vertices incident from an arbitrary vertex n given by $j = 4n-15, 4n-7, 4n, 4n+1$. Clearly, for $n \geq 4$, the out-degree of the vertex n is 4.

Similarly, the vertices incident to n are given by $i = \dfrac{n-1}{4}, \dfrac{n}{4}, \dfrac{n+7}{4}, \dfrac{n+15}{4}$. Thus, for $n \geq 4$,

$$\operatorname{indeg}(n) = \begin{cases} 1, & \text{if } n \equiv 0 \pmod 4, \\ 3, & \text{if } n \equiv 1 \pmod 4, \\ 0, & \text{if } n \equiv 2,3 \pmod 4. \end{cases}$$

Consequently, the out-degree and the in-degree sequences for the digraph $U^3_\infty\langle 3,4;1,3\rangle$ are $(2,3,3,4,4,4,\dots)$ and $(0,\dots,0,1,1,\dots,1,1,2,3,3,\dots)$ respectively.

Readers are left to discuss these properties for higher values of k, and find out if there are any similarities or patterns in them.

5 Conclusion and Scope

We have established the block matrix decomposition of k^{th} order slant Toeplitz operators. We have also discussed some relationships between the compressions of k^{th} order slant Hankel and k^{th} order slant Toeplitz operators. Further, we studied the graphs of k^{th} order slant Toeplitz operators on H^2 and drew some related graphs.

We can further study the above relations by using the compressions of these generalized slant Hankel and generalized slant Toeplitz operators to model spaces [10]. We can also discuss several properties like norm, spectrum, compactness etc., of the products of such truncated operators [6]. Hamiltonian properties of Toeplitz and directed Toeplitz graphs have been studied in [14] and [11]. We can further examine and analyze these properties for generalized slant Toeplitz operators.

Various structural and spectral properties along with hyponormality and isometric behaviour of slant Toeplitz operators on the Lebesgue space of the n-torus have been thoroughly discussed in [8]. By introducing the notion of slantification of a Hankel operator on the Hardy space of n-torus, hyponormality, isometric behaviour, co-isometric behaviour and compactness of these operators have also been studied [7]. Thus, we can also analyze various relations between the compressions of k^{th} order slant Hankel and k^{th} order slant Toeplitz operators on such spaces.

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Conflicts of Interest The authors declare no conflict of interest.

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